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Spin-Two Symmetric Spinors and Elastic Scattering of Massive Spin-Two Particles*

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Abstract

Some properties of the Hamiltonian description of free spin-two massive and massless particles are given emphasizing the connection with symmetric spinors having four free indices. In addition, the most general scattering amplitude is constructed for the elastic scattering of massive spin-two particles and scalar particles including the massless limit.

1. *Introduction*

Weaver, Hammer, and Good (1964) (WHG) have given a Hamiltonian formulation of the theory of a free particle and antiparticle with arbitrary mass and spin $S = 0, \frac{1}{2}, 1, \ldots$. The basis of their approach is to represent the spin S particle by the (S, O) and (O, S) representations of the homogeneous Lorentz group. The usefulness of these representations has been discussed from other points of view by Joos (1962) and Weinberg (1964), and a review has been given by Nelson and Good (1968).

To construct the Hamiltonians in WHG, the Foldy-Wouthuysen (FW) transformation (1950) was generalized to arbitrary spin, but in a way that is not unitary (see footnote 1) except for spin $\frac{1}{2}$. Later Weaver (1968) found a unitary transformation for spin-1 with many of the properties of the generalized FW transformation.

Investigating the description of WHG further, Weaver and Fradkin (1965) showed that the wavefunction, which has $2(2S + 1)$ components for spin S, was formed from the independent components of two symmetric spinors, each with $2S$ double-valued indices, and related by a spinor wave equation.

The purpose of this paper is to examine explicitly the spin-2 specialization of the general description including the connection with the symmetric spinors

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[1] By unitary is meant equality of the Hermitian conjugate and the inverse for operators.

and the massless limit. A further purpose is to construct the scattering amplitude consistent with Lorentz invariance and invariance with respect to space and time reflection and charge conjugation for the elastic scattering of massive spin-two particles and scalar particles, including the massless limit.

In contrast to conventional treatments, the massive spin-two particles will be represented by symmetric spinor field operators. This simplifies the construction, particularly the massless limit because no additional constraints (the spin-2 analogue of gauge invariance) need be imposed.

It should be emphasized that there are difficulties in taking the massless limit when one needs to deal with the nonlinear aspects of the spin two field found in nature, the gravitational field, but the difficulties do not appear in the present work (see, for example, van Dam and Veltman (1970), Boulware and Deser (1972) and van Nieuwenhuizen (1973)).

2. *Description of a Spin-Two Particle*

Following Weaver *et al.* (1964) the wavefunction $\psi(x)$, representing a particle and antiparticle of mass m and spin-2, is 10-dimensional and satisfies the wave equation

$$H\psi(x) = i\frac{\partial}{\partial t}\psi(x) \tag{2.1}$$

The units are $\hbar = C = 1$ and $X \equiv (\mathbf{X}, it)$. In terms of the momentum operator $\mathbf{P} \equiv -i\nabla$, the energy operator $E \equiv \sqrt{(\mathbf{P}.\mathbf{P} + m^2)}$ and the 10×10 matrices

$$\boldsymbol{\alpha} \equiv \tfrac{1}{2}\begin{pmatrix} \mathbf{S} & 0 \\ 0 & -\mathbf{S} \end{pmatrix}, \qquad \beta \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

(with $\mathbf{S}$ the representation of spin matrices with S_3 diagonal) the Hamiltonian operator H is given by

$$H/E = \tanh[4\tanh^{-1}(P/E)\boldsymbol{\alpha}.\hat{\mathbf{P}}] + \beta\,\mathrm{sech}[4\tanh^{-1}(P/E)\boldsymbol{\alpha}.\hat{\mathbf{P}}] \tag{2.3}$$

Note that H is a Hermitian matrix, and that $H^2 = E^2$ so that $\psi(x)$ satisfies the Klein-Gordon equation for mass m.

With respect to the homogeneous Lorentz group, $\psi(x)$ transforms as the direct sum of the (0, 2) and (2, 0) representations, corresponding respectively to symmetric spinors with 4 lower-dotted indices and symmetric spinors with 4 upper-undotted indices. In detail, let χ and φ be the symmetric spinors appropriate for spin two objects (see footnote 2). Then, up to factors which are Lorentz scalars, the wavefunction $\psi(x)$ and the symmetric spinors are related by

$$\begin{gathered}\psi_1 = x_{\dot{1}\dot{1}\dot{1}\dot{1}}, \quad \psi_2 = 2x_{\dot{1}\dot{1}\dot{1}\dot{2}}, \quad \psi_3 = \sqrt{(6)}x_{\dot{1}\dot{1}\dot{2}\dot{2}}, \quad \psi_4 = 2x_{\dot{1}\dot{2}\dot{2}\dot{2}}, \\ \psi_5 = x_{\dot{2}\dot{2}\dot{2}\dot{2}}, \quad \psi_6 = \varphi^{1111}, \quad \psi_7 = 2\varphi^{1112}, \quad \psi_8 = \sqrt{(6)}\varphi^{1122}, \\ \psi_9 = 2\varphi^{1222}, \quad \psi_{10} = \varphi^{2222}\end{gathered} \tag{2.4}$$

[2] A symmetric spinor with four indices has five independent components and is appropriate for describing a spin-two particle.

In terms of momentum spinors and second-quantized operator coefficients, the symmetric spinors are given by

$$\chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3P}{\sqrt{(2E)}}\frac{[E+m+\boldsymbol{\sigma}.\mathbf{P}]}{[2(E+m)]^2}{}_{\alpha_1\beta_1}[E+m+\boldsymbol{\sigma}.\mathbf{P}]_{\alpha_2\beta_3}\times$$
$$[E+m+\boldsymbol{\sigma}.\mathbf{P}]_{\alpha_3\beta_3}[E+m+\boldsymbol{\sigma}.\mathbf{P}]_{\alpha_4\beta_4}\sum_{k=-2}^{a} u_{\beta_1\beta_2\beta_3\beta_4}(\hat{\mathbf{P}},k)\times$$
$$[a(\mathbf{P},k)e^{\iota P.x} + (-1)^{a-k\dagger}b(\mathbf{P}_1-k)e^{-iP.x}] \qquad (2.5)$$

and a similar expression for $\varphi^{\alpha_1\alpha_2\alpha_3\alpha_4}(x)$ with $\boldsymbol{\sigma}.\mathbf{P}$ replaced everywhere by $-\boldsymbol{\sigma}.\mathbf{P}$. Here a and $b^\dagger$ are, respectively, the destruction and creation operators for particle and antiparticle; $\boldsymbol{\sigma}$ are the representation of Pauli matrices with σ_3 diagonal; k is the polarization quantum number, taking on values from -2 to 2 in integer steps, and $U(\hat{\mathbf{P}},k)$ are the momentum-polarization spinors appropriate for spin-two (see footnote 3). The Lorentz invariant scalar product $\mathbf{P}.x$ is $\mathbf{P}.\mathbf{x} - Et$. The momentum-polarization spinors have the property

$$[P\pm\boldsymbol{\sigma}.\mathbf{P}]_{\alpha_1\beta_1}[P\pm\boldsymbol{\sigma}.\mathbf{P}]_{\alpha_2\beta_2}[P\pm\boldsymbol{\sigma}.\mathbf{P}]_{\alpha_3\beta_3}[P\pm\boldsymbol{\sigma}.\mathbf{P}]_{\alpha_4\beta_4}U_{\beta_1\beta_2\beta_3\beta_4}(\hat{\mathbf{P}}.k)=0 \qquad (2.6)$$

unless $k = \pm 2$, respectively. This leads, in the limit that the particle mass goes to zero, to the following pair of symmetric spinors appropriate for describing a massless, spin-two particle and antiparticle

$$\chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(x) = \frac{1}{(2\pi)^{3/2}}\int d^3P(2P)^{3/2}U_{\alpha_1\alpha_2\alpha_3\alpha_4}(\hat{\mathbf{P}},2)[a(\mathbf{P},2)e^{iP.x} +$$
$$b^\dagger(\mathbf{P},-2)e^{-iP.x}] \qquad (2.7)$$

$$\varphi^{\alpha_1\alpha_2\alpha_3\alpha_4}(x) = \frac{1}{(2\pi)^{3/2}}\int d^3P(2P)^{3/2}U_{\alpha_1\alpha_2\alpha_3\alpha_4}(\hat{\mathbf{P}},-2)[a(\mathbf{P},-2)e^{iP.x} +$$
$$b^\dagger(\mathbf{P},2)e^{-iP.x}] \qquad (2.8)$$

Being interested in the massless limit of the spin-two theory leads one to consider, as well as equation (2.5), the spinor formed by operation on $\chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}$ with $P^{\gamma\dot{\alpha}}$ the spinor divergence operator defined by $P^{\gamma\dot{\alpha}} \equiv (i\boldsymbol{\sigma}\cdot\nabla + i(\partial/\partial t))\gamma\alpha$. One finds that $P^{\gamma\dot{\alpha}_1}\chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(x)$ is directly proportional to the first power of the mass of the particle with a coefficient that goes smoothly to a finite result in the limit of zero mass. The result is that both $P_{\alpha_1\dot{\gamma}}\varphi^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}$ and $P^{\gamma\dot{\alpha}_1}X_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\alpha_4}$ vanish smoothly as the mass of the particle goes to zero.

3. *Unitary, Spin-Two, Massless FW Operator*

In the limit of zero particle mass the spin-two Hamiltonian given in equation (2.3) becomes

[3] They are constructed from the eigenstates of the spin $\frac{1}{2}$ polarization operator with the usual rules for addition of angular momenta.

$$H_0 = \boldsymbol{\alpha}.\mathbf{P}[\tfrac{7}{3} - \tfrac{4}{3}(\boldsymbol{\alpha}.\mathbf{P})^2] + \beta P[1 - 4(\boldsymbol{\alpha}.\mathbf{P})^2][1 - (\boldsymbol{\alpha}.\mathbf{P})^2] \tag{3.1}$$

Since

$$(\boldsymbol{\alpha}.\mathbf{P})^2\,\psi(x) \xrightarrow{m\to 0} \psi(x) \tag{3.2}$$

because $(\boldsymbol{\alpha}.\hat{\mathbf{P}})^2$ is the square of the normalized, spin-two helicity operator, one sees that in the limit of zero particle mass.

$$H_0\psi(x) = \boldsymbol{\alpha}.\mathbf{P}\psi(x) \tag{3.3}$$

a simple result which, in fact, is true for all spins and zero masses. By contrast, in the limit of zero particle momentum (rest system) the general spin-two Hamiltonian assumes the simple form $m\beta$. The operator

$$\begin{aligned} U &\equiv \exp\left[\frac{5\pi}{4}\,\boldsymbol{\alpha}.\hat{\mathbf{P}}\beta + \frac{\pi}{4}(\boldsymbol{\alpha}\cdot\hat{\mathbf{P}})^3\beta\right] \\ &= \frac{1}{\sqrt{(2)}} + \left(1 - \frac{1}{\sqrt{(2)}}\right)[1 - 4(\boldsymbol{\alpha}.\hat{\mathbf{P}})^2][1 - (\boldsymbol{\alpha}.\hat{\mathbf{P}})^2] \\ &\quad + \sqrt{(2)}\beta[\boldsymbol{\alpha}.\hat{\mathbf{P}} + \tfrac{1}{6}\,\boldsymbol{\alpha}.\hat{\mathbf{P}}[1 - 4(\boldsymbol{\alpha}.\hat{\mathbf{P}})^2]] \end{aligned} \tag{3.4}$$

connects the two extreme forms of the spin-two Hamiltonian, i.e. $UH_0U^{-1} = P\beta$ and it is designated the spin-two, massless FW operator. It is unitary so ordinary scalar products retain their values under this transformation. The usefulness of this operator transformation is in the simple matrix forms of operators such as the Hamiltonian in the 'rest system' representation. It is also interesting that one can in a formal way transform a massless particle theory to a representation that is most appropriate for massive particles at rest.

4. *Spin-Two, Scalar Elastic Scattering*

Let g denote a massive spin-two particle and σ a scalar particle. Then, for the process

$$g_1(K_1) + \sigma_1(Q_1) \to g_2(K_2) + \sigma_2(Q_2) \tag{4.1}$$

let the four-momentum operator be written as K_1 when it operates on the g_1 field operator, Q_1 on σ_1, etc. so that the order of derivative factors in the $\mathscr{R}$-operator, defined in terms of the scattering operator S by

$$S = 1 + i\int d^4x\,\mathscr{R}(x) \tag{4.2}$$

may be disregarded. Invariance of the S-operator with respect to space-time translations leaves only three of the field operators with independent derivatives. They are chosen to be K_1, K_2 and the symmetric combination $Q \equiv \frac{1}{2}(Q_1 + Q_2)$. The σ-particles are described by the one-component, scalar field operators $\phi(1)$ and $\phi(2)$, which are assumed to be self-charge-conjugate. The massive spin-two field operators are $\chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)$, $\varphi^{\alpha_1\alpha_2\alpha_3\alpha_4}(1)$ and $\chi_{\alpha_1\alpha_2\alpha_3\alpha_4}(2)$, $\varphi^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(2)$, symmetric spinor pairs. Each pair is coupled by a fourth-order wave equation, equivalent to equation (2.1) when the Hamiltonian description is used. The symmetry of the spinors makes the number of independent component five, as seen in equation (2.4), appropriate for a spin-two object.

Use of the wave equations and the relation

$$A_{\alpha\dot{\beta}_1}B^{\alpha\dot{\beta}_2} + A^{\alpha\dot{\beta}_2}B_{\alpha\dot{\beta}_1} = -2A\,.\,B\delta_{\beta_1\beta_2} \tag{4.3}$$

true for any two form-vectors A and B, permit one to eliminate all terms in the $\mathscr{R}$-operator in favor of those with four or fewer derivatives. Space-inversion invariance further restricts the number of independent terms. The result for the Lorentz invariant, space-inversion covariant, charge conjugation invariant $\mathscr{R}$-operator for the elastic scattering of spin-two bosons and scalar bosons and all related processes ($g_2 \neq g_1$) is

$$\mathscr{R}(x) = \sum_{i=1}^{13} B_i[N_i + N_i^e]\phi(1)\phi(2) \tag{4.4}$$

where B_i are scalar functions of the independent derivatives, and N_i^e is the charge conjugate of N_i. In detail, the individual terms are (see footnote 4)

$$N_1 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)\varphi^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2)$$

$$N_2 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)Q^{\gamma_1\dot{\alpha}_1}Q^{\gamma_2\dot{\alpha}_2}Q^{\gamma_3\dot{\alpha}_3}Q^{\gamma_4\dot{\alpha}_4}\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)Q_{\gamma_1\dot{\alpha}_1}Q_{\gamma_2\dot{\alpha}_2}Q_{\gamma_3\dot{\alpha}_3}Q_{\gamma_4\dot{\alpha}_4}\varphi^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(2)$$

$$N_3 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)K_{2\gamma\dot{\alpha}_5}K_1^{\gamma\dot{\alpha}_1}\varphi^{\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4\dot{\alpha}_5}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)K_2^{\gamma_5\dot{\alpha}}K_{1\gamma_1\dot{\alpha}}\chi_{i_2\gamma_3\gamma_4\gamma_5}(2)$$

$$N_4 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)Q_{\gamma\dot{\alpha}_5}K_1^{\gamma\dot{\alpha}_1}\varphi^{\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4\dot{\alpha}_5}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)Q^{\gamma_5\dot{\alpha}}K_{1\gamma_1\dot{\alpha}}\chi_{\gamma_2\gamma_3\gamma_4\gamma_5}(2)$$

$$N_5 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)K_{2\gamma\dot{\alpha}_5}Q^{\gamma\dot{\alpha}_1}\varphi^{\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4\dot{\alpha}_5}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_5}(1)K_2^{\gamma_5\dot{\alpha}}Q_{\gamma_1\dot{\alpha}}\chi_{\gamma_2\gamma_3\gamma_4\gamma_5}(2)$$

$$N_6 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)K_1^{\gamma_1\dot{\alpha}_1}Q^{\gamma_2\dot{\alpha}_2}Q^{\gamma_3\dot{\alpha}_3}Q^{\gamma_4\alpha_4}\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)K_{1_{\gamma_1\dot{\alpha}_1}}Q_{\gamma_2\alpha_2}Q_{\gamma_3\dot{\alpha}_3}Q_{\gamma_4\dot{\alpha}_4}\varphi^{\alpha_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(2)$$

$$N_7 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)K_2^{\gamma_1\dot{\alpha}_1}Q^{\gamma_2\dot{\alpha}_2}Q^{\gamma_3\dot{\alpha}_3}Q^{\gamma_4\dot{\alpha}_4}\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)K_{2_{\gamma_1\dot{\alpha}_1}}Q_{\gamma_2\dot{\alpha}_2}Q_{\gamma_3\dot{\alpha}_3}Q_{\gamma_4\dot{\alpha}_4}\varphi^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(2)$$

$$N_8 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)K_1^{\gamma_1\dot{\alpha}_1}K_1^{\gamma_2\dot{\alpha}_2}Q^{\gamma_3\dot{\alpha}_3}Q^{\gamma_4\dot{\alpha}_4}\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)K_{1_{\gamma_1\dot{\alpha}_1}}K_{1_{\gamma_2\dot{\alpha}_3}}Q_{\gamma_3\dot{\alpha}_3}Q_{\gamma_4\dot{\alpha}_4}\varphi^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(2)$$

$$N_9 = \chi_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(1)K_2^{\gamma_1\dot{\alpha}_1}K_2^{\gamma_2\dot{\alpha}_3}Q^{\gamma_3\alpha_3}Q^{\gamma_4\alpha_4}\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)K_{2_{\gamma_1\dot{\alpha}_1}}K_{2_{\gamma_2\dot{\alpha}_2}}Q_{\gamma_3\dot{\alpha}_3}Q_{\gamma_4\dot{\alpha}_4}\varphi^{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3\dot{\alpha}_4}(2)$$

[4] The N_i are understood to be symmetrized in the usual way to avoid infinites in matrix elements. See, for example, the discussion in Sakurai J. J. (1964). *Invariance Principles and Elementary Particles*, p. 123, Princeton University Press, New Jersey.

$$N_{10} = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K_1^{\gamma_1\dot\alpha_1} K_2^{\gamma_2\dot\alpha_2} Q^{\gamma_3\dot\alpha_3} Q^{\gamma_4\dot\alpha_4} \chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1) K_{1_{\gamma_1\dot\alpha_1}} K_{2_{\gamma_2\dot\alpha_2}} Q_{\gamma_3\dot\alpha_3} Q_{\gamma_4\dot\alpha_4} \varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)$$

$$N_{11} = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K_1^{\gamma_1\dot\alpha_1} K_1^{\gamma_2\dot\alpha_2} K_2^{\gamma_3\dot\alpha_3} Q^{\gamma_4\dot\alpha_4} \chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1) K_{1_{\gamma_1\dot\alpha_1}} K_{1_{\gamma_2\dot\alpha_2}} K_{2_{\gamma_3\dot\alpha_3}} Q_{\gamma_4\dot\alpha_4} \varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)$$

$$N_{12} = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K_1^{\gamma_1\dot\alpha_1} K_2^{\gamma_2\dot\alpha_2} K_2^{\gamma_3\dot\alpha_3} Q^{\gamma_4\dot\alpha_4} \chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4} K_{1_{\gamma_1\dot\alpha_1}} K_{2_{\gamma_2\dot\alpha_2}} K_{2_{\gamma_3\dot\alpha_3}} Q_{\gamma_4\dot\alpha_4} \varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)$$

$$N_{13} = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K_1^{\gamma_1\dot\alpha_1} K_1^{\gamma_2\dot\alpha_2} K_2^{\gamma_3\dot\alpha_3} K_2^{\gamma_4\dot\alpha_4} \chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4} K_{1_{\gamma_1\dot\alpha_1}} K_{1_{\gamma_2\dot\alpha_2}} K_{2_{\gamma_3\dot\alpha_3}} K_{2_{\gamma_4\dot\alpha_4}} \varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2) \quad (4.5)$$

If the initial and final spin-two particles are identical, one has the following relations between the invariants and their charge conjugates

$$N_i^e = N_i, \qquad i = 1, 2, 3, 10, 13$$
$$N_i^e = N_{i+1}, \qquad i = 4, 6, 8, 11$$
$$N_i^e = N_{i-1}, \qquad i = 5, 7, 9, 12 \qquad (4.6)$$

This results in a simplified $\mathscr{R}$-operator with only nine independent terms. In detail, one has in this special case

$$\mathscr{R}(x) = \sum_{i=1}^{9} A_i M_i \phi(1)\phi(2) \qquad (4.7)$$

where (defining $K \equiv \frac{1}{2}(K_1 + K_2)$)

$$M_1 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1)\varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2)$$

$$M_2 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) Q^{\gamma_1\dot\alpha_1} Q^{\gamma_2\dot\alpha_2} Q^{\gamma_3\dot\alpha_3} Q^{\gamma_4\dot\alpha_4} \chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1) Q_{\gamma_1\dot\alpha_1} Q_{\gamma_2\dot\alpha_2} Q_{\gamma_3\dot\alpha_3} Q_{\gamma_4\dot\alpha_4} \varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)$$

$$M_3 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K^{\gamma\dot\alpha_1} Q_{\gamma\dot\alpha_5} \varphi^{\dot\alpha_2\dot\alpha_3\dot\alpha_4\dot\alpha_5}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1) K_{\gamma_1\dot\alpha} Q^{\gamma_5\dot\alpha} \chi_{\gamma_2\gamma_3\gamma_4\gamma_5}(2)$$

$$M_4 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K^{\gamma_1\dot\alpha_1} Q^{\gamma_2\dot\alpha_2} Q^{\gamma_3\dot\alpha_3} Q^{\gamma_4\dot\alpha_4} \chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1) K_{\gamma_1\dot\alpha_1} Q_{\gamma_2\dot\alpha_2} Q_{\gamma_3\dot\alpha_3} Q_{\gamma_4\dot\alpha_4} \varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)$$

$$M_5 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K_1^{\gamma\dot\alpha_1} K_{2_{\gamma\dot\alpha_5}} \varphi^{\dot\alpha_2\dot\alpha_3\dot\alpha_4\dot\alpha_5}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1) K_{1_{\gamma_1\dot\alpha}} K_2^{\gamma_5\dot\alpha} \chi_{\gamma_2\gamma_3\gamma_4\gamma_5}(2)$$

$$M_6 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K^{\gamma_1\dot\alpha_1} K^{\gamma_2\dot\alpha_2} Q^{\gamma_3\dot\alpha_3} Q^{\gamma_4\dot\alpha_4} \chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1) K_{\gamma_1\dot\alpha_1} K_{\gamma_2\dot\alpha_2} Q_{\gamma_3\dot\alpha_3} Q_{\gamma_4\dot\alpha_4} \varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)$$

$$M_7 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1) K_1^{\gamma_1\dot\alpha_1} K_2^{\gamma_2\dot\alpha_2} Q^{\gamma_3\dot\alpha_3} Q^{\gamma_4\dot\alpha_4} \chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1) K_{1_{\gamma_1\dot\alpha_1}} K_{2_{\gamma_2\dot\alpha_2}} Q_{\gamma_3\dot\alpha_3} Q_{\gamma_4\dot\alpha_4} \varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)$$

$$M_8 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1)K^{\gamma_1\dot\alpha_1}K^{\gamma_2\dot\alpha_2}K^{\gamma_3\dot\alpha_3}Q^{\gamma_4\dot\alpha_4}\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}K_{\gamma_1\dot\alpha_1}K_{\gamma_2\dot\alpha_2}K_{\gamma_3\dot\alpha_3}Q_{\gamma_4\dot\alpha_4}\varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)$$

$$M_9 = \chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1)K^{\gamma_1\dot\alpha_1}K^{\gamma_2\dot\alpha_2}K^{\gamma_3\dot\alpha_3}K^{\gamma_4\dot\alpha_4}\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)K_{\gamma_1\dot\alpha_1}K_{\gamma_2\dot\alpha_2}K_{\gamma_3\dot\alpha_3}K_{\gamma_4\dot\alpha_4}\varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2) \quad (4.8)$$

The matrix elements of $\mathscr{R}$ integrated over all space-time are proportional to the usual T-matrix elements, and so all the usual properties of the T-matrix elements apply here.

As discussed in Section 2 the symmetric spinors go smoothly to the appropriate massless limit, and their derivatives are proportional to the particle mass. So, as the particle mass goes to zero M_3 and M_4 vanish as m, M_5, M_6 and M_7 as m^2, M_8 as m^3 and M_9 as m^4. It is important to note that there are no relations that the invariant amplitudes A_i are required to satisfy in this limit. The resulting $\mathscr{R}$-operator for spin-two "Compton scattering" is

$$\mathscr{R}(x) = A_1\{\chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1)\varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2) + \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)\,\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2)\}\phi(1)\phi(2)$$
$$+ A_2\{\chi_{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(1)Q^{\gamma_1\dot\alpha_1}Q^{\gamma_2\dot\alpha_2}Q^{\gamma_3\dot\alpha_3}Q^{\gamma_4\dot\alpha_4}\chi_{\gamma_1\gamma_2\gamma_3\gamma_4}(2)$$
$$+ \varphi^{\gamma_1\gamma_2\gamma_3\gamma_4}(1)Q_{\gamma_1\dot\alpha_1}Q_{\gamma_2\dot\alpha_2}Q_{\gamma_3\dot\alpha_3}Q_{\gamma_4\dot\alpha_4}\varphi^{\dot\alpha_1\dot\alpha_2\dot\alpha_3\dot\alpha_4}(2)\}\phi(1)\phi(2) \quad (4.9)$$

To make the connection with the Hamiltonian formulation, one notes that

$$\bar\psi(2)\psi(1) \equiv \psi^\dagger(2)\beta\psi(1)$$
$$= \varphi^{\dot1\dot1\dot1\dot1}(2)\chi_{\dot1\dot1\dot1\dot1}(1) + 4\varphi^{\dot1\dot1\dot1\dot2}(2)\chi_{\dot1\dot1\dot1\dot2}(1) + 6\varphi^{\dot1\dot1\dot2\dot2}(2)\chi_{\dot1\dot1\dot2\dot2}(1)$$
$$+ 4\varphi^{\dot1\dot2\dot2\dot2}(2)\chi_{\dot1\dot2\dot2\dot2}(1) + \varphi^{\dot2\dot2\dot2\dot2}(2)\chi_{\dot2\dot2\dot2\dot2}(1) + \chi_{1111}(2)\varphi^{1111}(1)$$
$$+ 4\chi_{1112}(2)\varphi^{1112}(1) + 6\chi_{1122}(2)\varphi^{1122}(1) + 4\chi_{1222}(2)\varphi^{1222}(1)$$
$$+ \chi_{2222}(2)\varphi^{2222}(1) \quad (4.10)$$

and that this expression is identical to M_1 taking account of the symmetry of the spinors to combine terms. Furthermore, up to an overall scalar factor

$$M_2 \sim \bar\psi(2)Q_\mu Q_\nu Q_\rho Q_\sigma \gamma_{\mu\nu\rho\sigma}\gamma(1) \quad (4.11)$$

where $\gamma_{\mu\nu\rho\sigma}$ are the spin two covariantly defined matrices, the generalization of Dirac's gamma matrices (4), (Nelson and Good, Jr., 1968).

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